ACADEMIC PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Sound and Vibration 260 (2003) 637–651

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

# Torsional vibration of multi-step non-uniform rods with various concentrated elements

Q.S. Li*

*Department of Building and Construction, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong*

Received 18 September 2001; accepted 3 May 2002

**Abstract**

An analytical approach and exact solutions for the torsional vibration of a multi-step non-uniform rod carrying an arbitrary number of concentrated elements such as rigid disks and with classical or non-classical boundary conditions is presented. The exact solutions for the free torsional vibration of non-uniform rods whose variations of cross-section are described by exponential functions and power functions are obtained. Then, the exact solutions for more general cases, non-uniform rods with arbitrary cross-section, are derived for the first time. In order to simplify the analysis for the title problem, the fundamental solutions and recurrence formulas are developed. The advantage of the proposed method is that the resulting frequency equation for torsional vibration of multi-step non-uniform rods with arbitrary number of concentrated elements can be conveniently determined from a homogeneous algebraic equation. As a consequence, the computational time required by the proposed method can be reduced significantly as compared with previously developed analytical procedures. A numerical example shows that the results obtained from the proposed method are in good agreement with those determined from the finite element method (FEM), but the proposed method takes less computational time than FEM, illustrating the present methods are efficient, convenient and accurate.


## 1. Introduction

The torsional vibration problems of structural members and machine parts carrying various concentrated elements, such as a shaft carrying several rigid disks, are always encountered in engineering practices. Hence, determination of natural frequencies and mode shapes of such systems subjected to twisting moments is necessary in the design of certain structural members or machine parts.

*Corresponding author. Tel.: +852-2784-4677; fax: +852-2788-7612.
*E-mail address:* bcqsli@cityu.edu.hk (Q.S. Li).

0022-460X/03/$ - see front matter 
doi:10.1016/S0022-460X(02)01010-6

The problem for determining the dynamic characteristics of a uniform twisting rod with circular cross-section is easily solved (e.g., Refs. [1–3]). But a concentrated element attached to a uniform rod increases the difficulty to solve the problem analytically. Although most of the approaches presented in the literature may be extended to solve the eigenvalue problem for a uniform rod carrying various concentrated elements, they are not easily implemented because of the complexity of the mathematical expressions. For this reason, the total number of concentrated elements reported in the literature (e.g., Ref. [4]) is usually less than two. Therefore, there is a need to propose an efficient analytical method to determine the natural frequencies and mode shapes of a twisting rod carrying an arbitrary number of concentrated elements.

Much work has been done to investigate the torsional vibration of uniform rods, however, dynamic of non-uniform twisting rods with an arbitrary number of concentrated elements received less attention in the past. It should be mentioned that structural members and machine parts with variable cross-section are frequently used in engineering practices to optimize the distributions of weight and strength. In fact, most of the twisting rods used in engineering practices are not uniform. For example, as pointed out by Pouyet and Lataillade ([5]), in respect of the geometry of a shaft, even if the main part of the shaft is uniform, this may not be the case for the ends: there may be a ramp shoulder, an overflow cone or a conical shaft end. One also can find intermediate portions with variable cross-section (e.g., a fillet) connecting two pieces of uniform shafts. If a rod has variable cross-section, then the displacements, in general, are not proportional to the radial distance from the axis of twist, especially, if the cross-section is not circular, there is some warping of the cross-sectional plane associated with torsional motion. In order to simplify the calculation it is often assumed that the shape of the cross-sectional area and non-uniformity of the rod are such that the motion can be regarded as rotation of the cross-sectional plane as a whole and without warping [1,6]. Under this assumption Pouyet and Lataillade ([5]) proposed an analytical approach for determining the natural frequencies of rotating shafts of variable cross-section. But, in their study, the general case that the distribution of the torsional stiffness is not proportional to that of the mass polar moment of inertia was not considered.

Literature review indicates that the authors of previous studies have generally directed their investigation to special functions for describing the distributions of torsional stiffness and mass polar moment of inertia of a non-uniform twisting rod in order to derive the closed-form solutions. The existing analytical solutions (e.g., Refs. [4–6]) are limited to several types of non-uniform rods only. It is noted that the analytical solutions for free torsional vibration of a non-uniform rod with arbitrary distribution of torsional stiffness or mass polar moment of inertia and carrying an arbitrary number of concentrated elements have not been obtained in the past. In this paper, a successful attempt is made to present an efficient analytical method and exact solutions for the torsional vibration of non-uniform columns carrying an arbitrary number of concentrated elements. The exact solutions for the free torsional vibration of non-uniform rods whose variations of cross-section are described by exponential functions and power functions are obtained. Then, the exact solutions for more general cases, non-uniform rods with arbitrary cross-section, are also derived for the first time. As a consequence, the analytical solutions previously obtained by Blevins ([4]), Pouyet and Lataillade ([5]), Li et al. ([6]), etc. for special types of non-uniform rods actually result as special cases of the present exact solutions for torsional vibration of non-uniform rods with arbitrary cross-section. In order to simplify the analysis for the torsional vibration of multi-step non-uniform rods with an arbitrary number of concentrated elements, the fundamental solutions and recurrence formulas are developed. The advantage of the proposed method is that the resulting frequency equation expressed in terms of the fundamental solutions for torsional vibration of multi-step non-uniform rods with an arbitrary number of steps and concentrated elements can be conveniently determined from a homogeneous algebraic equation. As a result, the computational time required by the proposed method can be reduced significantly as compared with previously developed analytical procedures. A numerical example shows that the results obtained from the proposed method are in good agreement with those determined from the finite element method (FEM), but the proposed methods takes less computational time than FEM, illustrating the present methods are efficient, convenient and accurate.

Apart from several analytical methods (e.g., Refs. [4–7]) for analyzing limited classes of non-uniform twisting rods, many approximate and numerical methods (e.g., Refs. [8–11]) have been developed. In the absence of the exact solutions presented in this paper, the title problem may be solved using approximate or numerical methods. However, the present exact solutions could provide adequate insight into the physics of the problem and can be easily implemented. On the other hand, the availability of the exact solutions will help in examining the accuracy of the approximate or numerical solutions. Therefore, it is always desirable to obtain the exact solutions to such problem.

## 2. Theory

A multi-step non-uniform rod carrying an arbitrary number of concentrated elements such as rigid disks is shown in Fig. 1. The material of the rod is assumed to be elastic, homogeneous and isotropic, the shape of the cross-sectional area and the non-uniformity of the rod are such that the motion can be regarded as rotation of the cross-sectional plane as a whole and without warping. Letting the number of the concentrated elements located in the $i$th step rod be $n_i$, and the $n_i$ concentrated elements be located at sections $x_{i1}, x_{i2}, \ldots, x_{in_i}$ such that $0 < x_{i1} < x_{i2} < \cdots < x_{in_i} < l_i$, $l_i$ is the length of the $i$th step rod and the origin of co-ordinate system is set at the left end of this step rod. Because the difference between a rod with concentrated elements and the rod without the concentrated elements is that the twisting torque at the $ij$th ($j = 1, 2, \ldots, n_i$) section has a jump caused by the $ij$th concentrated element, in order to study the dynamic characteristics of a rod

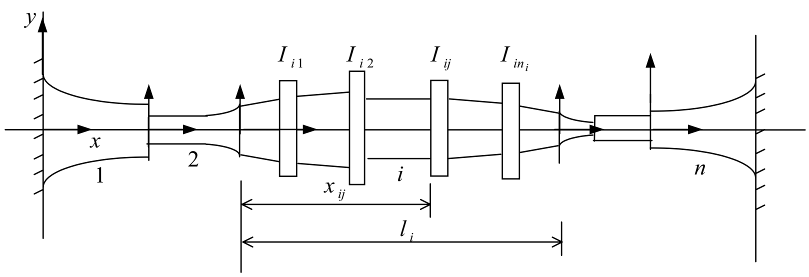


Fig. 1. A multi-step non-uniform rod with concentrated elements (disks). Note: The disks attached to the other step rods (except the $i$th step rod) are not shown in Fig. 1.

with concentrated elements it is necessary to understand those of the rod without concentrated elements first.

The governing differential equation for torsional mode shape function, $\Theta_i(x)$, of the $i$th step rod without concentrated element can be written as [3]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[GJ_i(x)\frac{\mathrm{d}\Theta_i(x)}{\mathrm{d}x}\right]+\omega^2 I_i(x)\Theta_i(x)=0, \tag{1}$$

where $GJ_i(x)$ and $I_i(x)$ are the torsional stiffness and mass polar moment of inertia per unit length, respectively, $\omega$ is the circular natural frequency, $G$ is the shear modulus, $J_i(x)$ is the polar moment of inertia of the cross-section of the $i$th step rod.

The general solution of Eq. (1) can be expressed as

$$\Theta x) = C_{i1}S_{i1}(x) + C_{i2}S_{i2}(x), \tag{2}$$

where $S_{ij}(x)$ and $C_{ij}$ $(j = 1, 2)$ are the linearly independent solutions and integral constants of Eq. (1), respectively.

It is evident that $S_{ij}(x)$ $(j = 1,\ 2)$ are dependent on the expression of $J_i(x)$ since $I_i(x)$, in general, is equal to $\rho J_i(x)$, $\rho$ is the mass intensity of material. Obviously, it is difficult to derive the analytical solution of Eq. (1) for general cases, because $J_i(x)$ in the equation varies with $x$. However, the analytical solution may be obtained by means of reasonable selections for $J_i(x)$. Hence, the following several cases of $J_i(x)$ which cover many types of non-uniform structural and mechanical members are considered in this paper.

*Case* 1: The distribution of polar moment of inertia of cross-section is described by a power function

$$J_i(x) = \alpha_i(1+\beta_i x)^{\gamma_i}, \tag{3}$$

where $\alpha_i$, $\beta_i$ and $\gamma_i$ are constants that can be determined by use of the real values of $J_i(x)$ at several control sections.

Substituting Eq. (3) into Eq. (1) results in

$$\frac{\mathrm{d}^2\Theta_i(\xi)}{\mathrm{d}\xi^2}+\frac{\gamma_i}{\xi}\frac{\mathrm{d}\Theta_i(\xi)}{\mathrm{d}\xi}+\lambda_i^2\Theta(\xi)=0, \tag{4}$$

where

$$\xi = 1+\beta_i x, \qquad \lambda_i^2=\frac{\rho\omega^2}{G\beta_i^2}. \tag{5}$$

Introducing the following functional transform for Eq. (4):

$$\Theta_i(\xi) = (\lambda_i\xi)^{v_i}Z_i, \qquad v_i^2=\frac{1-\gamma_i}{2} \tag{6}$$

one obtains a Bessel's equations as follows:

$$\frac{\mathrm{d}^2 Z_i}{\mathrm{d}\xi^2}+\frac{1}{\xi}\frac{\mathrm{d}Z_i}{\mathrm{d}\xi}+\left(1-\frac{v_i}{\xi^2}\right)Z_i=0. \tag{7}$$

The linearly independent solutions are

$$S_{i1}(x) = (1+\beta_i x)^{v_i}J_{v_i}[\lambda_i(1+\beta x)], \tag{8}$$

$$S_{i2}(x) = \begin{cases} (1+\beta_i x)^{v_i} J_{-v_i}[\lambda_i(1+\beta x)], & v_i = \text{a non-integer}, \\ (1+\beta_i x)^{v_i} Y_{v_i}[\lambda_i(1+\beta x)], & v_i = \text{an integer}. \end{cases} \tag{9}$$

If $\gamma_i = 1$, then $v_i = 0$.

If $\gamma_i = 2$, then $v_i = -\frac{1}{2}$, for this case the Bessel functions in the above solutions are reduced to trigonometrical functions and the linearly independent solutions for this case are

$$S_{i1}(x) = \frac{1}{1+\beta_i x}\sin[\lambda_i(1+\beta_i x)], \tag{10}$$

$$S_{i2}(x) = \frac{1}{1+\beta_i x}\cos[\lambda_i(1+\beta_i x)]. \tag{11}$$

*Case* 2: The distribution of polar moment of inertia of cross-section is described by an exponential function

$$J_i(x) = \alpha_i \mathrm{e}^{-\beta_i x/L_i}, \tag{12}$$

where $\alpha_i$, $\beta_i$ are constants that can be determined by use of the real values of $J_i(x)$ at several control sections.

Substituting Eq. (12) into Eq. (1) one obtains

$$\frac{\mathrm{d}^2\Theta_i(x)}{\mathrm{d}x^2} - \frac{\beta_i}{L_i}\frac{\mathrm{d}\Theta_i(x)}{\mathrm{d}x} + \lambda_i^2\Theta_i(x) = 0, \tag{13}$$

where the expression of $\lambda_i^2$ is the same as Eq. (5).

The solutions of Eq. (13) for $\beta_i^2/L_i^2 - 4\lambda_i^2 < 0$ are

$$S_{i1}(x) = e^{\beta_i x/2L_i}\sin\frac{\mu_i x}{L_i}, \tag{14}$$

$$S_{i2}(x) = \mathrm{e}^{\beta_i x/2L_i}\cos\frac{\mu_i x}{L_i}, \tag{15}$$

where

$$\mu_i^2 = \frac{L_i^2}{4}\left(4\lambda_i^2 - \frac{\beta_i^2}{L_i^2}\right). \tag{16}$$

If $\beta_i^2/L_i^2 - 4\lambda_i^2 \geqslant 0$, then $\Theta_i(x) = 0$, this case corresponds to the static status of the rod.

*Case* 3: The distribution of the torsional stiffness or the mass polar moment of inertia is arbitrary.

As reviewed previously, the analytical solutions for free torsional vibration of a non-uniform rod with arbitrary distribution of torsional stiffness or mass polar moment of inertia have not been obtained in the past. In this paper, a successful attempt is made to solve this challenging problem. It is assumed that

$$\Theta_i(x) = \Theta_i(\varsigma), \qquad GJ_i(x) = \text{ arbitrary function},$$

$$I_i(x) = [GJ_i(x)]^{-1}p(\varsigma), \quad \varsigma = \int [GJ_i(x)]^{-1}\,\mathrm{d}x. \tag{17}$$

or

$$\Theta_i(x) = \Theta_i(\varsigma), \qquad I_i(x) = \text{ arbitrary function,}$$
$$GJ_i(x) = I_i^{-1}(x)p(\varsigma), \qquad \varsigma = \int [GJ_i(x)]^{-1}\,\mathrm{d}x.$$

It can be seen from Eq. (17) that we consider a general case here, in which the distribution of torsional stiffness is described by an arbitrary function, and the variation of mass polar moment of inertia is expressed as a functional relation with the torsional stiffness and vice versa.

Substituting Eq. (18) into Eq. (1) results in

$$\frac{\mathrm{d}^2\Theta_i(x)}{\mathrm{d}\varsigma^2} + \omega^2 p_i(\varsigma)\Theta_i(\varsigma) = 0. \tag{18}$$

It is noted that in Eq. (17) one of the expressions of $GJ_i(x)$ and $I_i(x)$ is an arbitrary function, $\varsigma$ is a function of $x$, and $p_i(\varsigma)$ is a functional expression. Hence, a solution of Eq. (18) actually represents a class of solutions of Eq. (1). On the other hand, the use of Eq. (17) can eliminate the derivative of the first order in the governing differential equation for free torsional vibration of a non-uniform rod. It is easier, in general, to solve the differential equation with variable coefficients of the second order without the derivative of the first order than to solve that with the derivative of the first order. Thus, the introduction of the functional transformations given in Eq. (17) can simplify the torsional vibration analysis of non-uniform rods and obtain the closed-form solutions for such problems. Therefore, it is decided to derive the analytical solutions of Eq. (18) here. The solution processes for several important cases are given below:

1.
$$p_i(\varsigma) = (a_i + b_i\varsigma)^{c_i} \tag{19}$$

The linearly independent solutions of Eq. (18) are

$$S_{i1}(\varsigma) = (a_i + b_i\varsigma)^{1/2} J_{v_i}[\bar{\alpha}_i(a_i + b_i\varsigma)^{1/2v_i}], \tag{20}$$

$$S_{i2}(\varsigma) = \begin{cases} (a_i + b_i\varsigma)^{1/2} J_{-v_i}[\bar{\alpha}_i(a_i + b_i\varsigma)^{1/2}], & v_i = \text{ a non-integer,} \\ (a_i + b_i\varsigma)^{1/2} Y_{v_i}[\bar{\alpha}_i(a_i + b_i\varsigma)^{1/2v_i}], & v_i = \text{ an integer,} \end{cases} \tag{21}$$

where

$$\bar{\alpha}_i = \frac{2v_i\omega^2}{|b_i|}, \qquad v_i = \frac{1}{c_i + 2}. \tag{22}$$

If $c_i = -2$, then $v_i = \infty$, the above solutions are not valid for this case. When $c_i = -2$, substituting Eq. (19) into Eq. (18) leads to an Euler equation, its solutions are

$$\left.\begin{aligned} S_{i1}(\varsigma) &= (a_i + b_i\varsigma)^{1/2}\sin[\tilde{\alpha}_i \ln(a_i + b_i\varsigma)] \\ S_{i2}(\varsigma) &= (a_i + b_i\varsigma)^{1/2}\cos[\tilde{\alpha}_i \ln(a_i + b_i\varsigma)] \end{aligned}\right\} \text{ for } 4\omega^2 - b_i^2 > 0, \tag{23}$$

or

$$S_{i1}(\varsigma) = (a_i + b_i\varsigma)^{1/2+\tilde{\alpha}_i} \qquad \text{for } 4\omega^2 - b_i^2 < 0, \qquad S_{i2}(\varsigma) = (a_i + b_i\varsigma)^{1/2-\tilde{\alpha}_i} \tag{24}$$

or

$$S_{i1}(\varsigma) = (a_i + b_i\varsigma)^{1/2} \qquad \text{for } 4\omega^2 - b_i^2 = 0, \qquad S_{i2}(\varsigma) = (a_i + b_i\varsigma)^{1/2}\ln(a_i + b_i\varsigma) \tag{25}$$

where

$$\tilde{\alpha}_i = \frac{\left|4\omega^2 - b_i^2\right|^{1/2}}{2|b_i|}. \tag{26}$$

2.
$$p_i(\varsigma) = a_i(1 + b_i\varsigma)^{c_i} \tag{27}$$

This case is an alteration of Case 1. The solutions of Eq. (18) for this case are

$$S_{i1}(\varsigma) = (1 + b_i\varsigma)^{1/2} J_{v_i}[\bar{\lambda}_i(1 + b_i\varsigma)^{1/2v_i}] \tag{28}$$

$$S_{i2}(\varsigma) = \begin{cases} (1 + b_i\varsigma)^{1/2} J_{-v_i}[\bar{\lambda}_i(1 + b_i\varsigma)^{1/2v_i}], & v_i = \text{a non-integer}, \\ (1 + b_i\varsigma)^{1/2} Y_{v_i}[\bar{\lambda}_i(1 + b_i\varsigma)^{1/2v_i}], & v_i = \text{an integer}, \end{cases} \tag{29}$$

where

$$\bar{\lambda}_i = \frac{2\omega v_i a_i^{1/2}}{|b_i|}, \qquad v_i = \frac{1}{c_i + 2}. \tag{30}$$

If $c_i = -2$, then $v_i = \infty$. The solutions for this special case are similar to those given in Eqs. (23)–(25).

3.
$$p_i(\varsigma) = a_i(\varsigma^2 + b_i)^{-2}, \quad a_i > 0, b_i > 0. \tag{31}$$

The solutions of Eq. (18) for this case are

$$S_{i1}(\varsigma) = (\varsigma^2 + b_i)^{1/2} \sin\xi, \tag{32}$$

$$S_{i2}(\varsigma) = (\varsigma^2 + b_i)^{1/2} \cos\xi, \tag{33}$$

where

$$\xi = \left(\frac{a_i\omega^2 + b_i}{b_i}\right)^{1/2} \arctan\frac{\varsigma}{b_i^{1/2}}. \tag{34}$$

4.
$$p_i(\varsigma) = a_i(\varsigma^2 - b_i)^{-2}, \quad a_i > 0, b_i > 0. \tag{35}$$

The solutions of Eq. (18) for this case are

$$S_{i1}(\varsigma) = (b_i - \varsigma^2)^{1/2} \sin \xi, \tag{36}$$

$$S_{i2}(\varsigma) = (b_i - \varsigma^2)^{1/2} \cos \xi, \tag{37}$$

where

$$\xi = \frac{1}{2}\left(\frac{a_i\omega^2 - b_i^2}{b_i}\right)^{1/2} \ln\frac{b_i^{1/2} + \varsigma}{b_i^{1/2} - \varsigma}. \tag{38}$$

5.
$$p_i(\varsigma) = a_i e^{b_i\varsigma} - c_i. \tag{39}$$

The solutions for this case are

$$S_{i1}(\varsigma) = J_{v_i}\left(\alpha_i e^{b_i\varsigma/2}\right) \tag{40}$$

$$S_{i2}(\varsigma) = \begin{cases} J_{-v_i}(\alpha_i e^{b_i\varsigma/2}), & v_i = \text{a non-integer}, \\ Y_{v_i}(\alpha_i e^{b_i\varsigma/2}), & v_i = \text{an integer}, \end{cases} \tag{41}$$

where

$$\alpha_i = \frac{2\omega a_i^{1/2}}{|b_i|}, \qquad v_i = \frac{2\omega c_i^{1/2}}{|b_i|}. \tag{42}$$

If $c_i = 0$, then $v_i = 0$.
If $b_i = c_i = 0$, then

$$S_{i1}(\varsigma) = \sin(a_i^{1/2}\omega\varsigma), \tag{43}$$

$$S_{i2}(\varsigma) = \cos(a_i^{1/2}\omega\varsigma). \tag{44}$$

In order to simplify the analysis for the title problem, based on the derived linearly independent solutions $S_{i1}(x)$ and $S_{i2}(x)$ presented above, two linearly independent solutions, denoted by $\bar{S}_{i1}(x)$ and $\bar{S}_{i2}(x)$, which are called the fundamental solutions in this paper, are chosen such that they satisfy the following normalization conditions at the origin of the co-ordinate system:

$$\begin{bmatrix} \bar{S}_{i1}(0) & \bar{S}'_{i1}(0) \\ \bar{S}_{i2}(0) & \bar{S}'_{i2}(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{45}$$

$\bar{S}_{i1}(x)$ and $\bar{S}_{i2}(x)$ can be easily constructed by

$$\begin{bmatrix} \bar{S}_{i1}(x) \\ \bar{S}_{i2}(x) \end{bmatrix} = \begin{bmatrix} S_{i1}[\varsigma(0)] & S'_{i1}[\varsigma(0)] \\ S_{i2}[\varsigma(0)] & S'_{i2}[\varsigma(0)] \end{bmatrix}^{-1} \begin{bmatrix} S_{i1}[\varsigma(x)] \\ S_{i2}[\varsigma(x)] \end{bmatrix}. \tag{46}$$

The primes in Eqs. (45) and (46) indicate differentiation with respect to the co-ordinate variable $x$.

The compatibility of the twisting deformation of the rod at location of the $ij$th concentrated element requires that

$$\Theta_i^R(x_{ij}) = \Theta_i^L x_{ij}, \tag{47}$$

$$GJ_i(x_{ij})\Theta_i'^R(x_{ij}) = GJ_i(x_{ij})\Theta_i'^L(x_{ij}) - \omega^2 I_{ij}\Theta_i^L(x_{ij}), \tag{48}$$

where the superscripts "$L$" and "$R$" represent the left side and right side of the section $x_{ij}$, respectively, $I_{ij}$ denotes the mass polar moment of inertia of the $ij$th concentrated element attached at the section $x_{ij}$ of the $i$th step rod, and the second term on the right side of Eq. (48) represents a jump of the internal twisting moment which indicates the presence of an inertia moment caused by the $ij$th concentrated element.

Using the fundamental solutions developed in this paper and Eqs. (47) and (48), one can obtain the mode shape function of torsional vibration of the $i$th step rod as follows:

$$\Theta_i(x) = \Theta_i(0)\bar{S}_{i1}(x) + \Theta_i'(0)\bar{S}_{i2}(x) - \sum_{j=1}^{n_i}\frac{\omega^2 I_{ij}\Theta_i(x_{ij})}{GJ_i(x_{ij})}\bar{S}_{i2}(x - x_{ij})H(x - x_{ij}). \tag{49}$$

The last term of the above equation represents the jumps of the internal twisting moment at $x_{ij}$ ($j = 1, 2, \ldots, \mathrm{n}_i$).

It is necessary to point out that the mode shape function of the segment, $x \in [x_{ij}, x_{i(j+1)}]$, is different from those of other segments of the $i$th step rod. But the Heaviside function $\mathrm{H}(x - x_{ij}) = 0$ if $x < x_{ij}$, therefore, the expression of $\Theta_i(x)$ given in Eq. (49), is suitable for the whole $i$th step rod.

It can be seen from Eq. (49) that the main advantage of using the fundamental solutions developed in this paper is that the mode shape functions for all segments considering the jumps of the internal twisting moment at the sections where the concentrated elements are attached to are easily expressed in terms of the fundamental solutions, and hence, the frequency equation is conveniently established by using such solutions.

The twisting angle and the internal twisting moment of a cross-section at all the common interfaces of two neighboring step rods are required to be continuous, i.e.,

$$\Theta_i(0) = \Theta_{i-1}(l_{i-1}), \tag{50}$$

$$M_i(0) = M_{i-1}(l_{i-1}). \tag{51}$$

If there is a concentrated element such as a rigid disk, its mass polar moment of inertia denoted by $I_{i-1}$, attached at the left end of the $i$th step rod (Fig. 2), we have

$$\Theta_i(0) = \Theta_{i-1}(l_{i-1}), \tag{52}$$

$$M_i(0) = M_{i-1}(l_{i-1}) - \omega^2 I_{i-1}\Theta_{i-1}(l_{i-1}). \tag{53}$$

Substituting Eqs. (52) and (53) into Eq. (49), one obtains

$$\begin{aligned}\Theta_i(x) = {} & \Theta_{i-1}(l_{i-1})\bar{S}_{i1}(x) + \frac{1}{GJ_i(0)}[M_{i-1}(l_{i-1}) - \omega^2 I_{i-1}\Theta_{i-1}(l_{i-1})]\bar{S}_{i2}(x) \\ & - \sum_{j=1}^{n_i}\frac{\omega^2 I_{ij}\Theta_i(x_{ij})}{GJ_i(x_{ij})}\bar{S}_{i2}(x - x_{ij})\mathrm{H}(x - x_{ij}).\end{aligned} \tag{54}$$

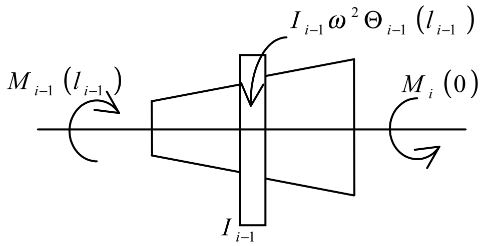


Fig. 2. The twisting moments at the neighboring region of a common interface.

This is a recurrence formula. Using this formula and $\Theta_1(x)$ which includes only one unknown initial parameter for any type of support condition at the left end of the first step rod, one can determine the mode shape function of the $i$th step rod ($i = 2, 3, \ldots, n$) which also includes the same unknown parameter as $\Theta_1(x)$ has.

The frequency equation for torsional vibration of a multi-step non-uniform beam carrying various concentrated elements may be obtained based on the specified boundary conditions as follows:

(1) *Two fixed ends*: The boundary conditions associated with the torsional vibration of a rod for this case are

$$\Theta_1(0) = 0, \tag{55}$$

$$\Theta_n(l_n) = 0. \tag{56}$$

Applying Eq. (55) to Eq. (54), we have

$$\Theta_1(x) = \frac{M_1(0)}{GJ_1(0)}\bar{S}_{12}(x) - \sum_{j=1}^{n_1}\frac{\omega^2 I_{1j}\Theta_1(x_{1j})}{GJ_1(x_{1j})}\bar{S}_{12}(x - x_{1j})\mathrm{H}(x - x_{1j}), \tag{57}$$

where $I_{1j}$ ($j = 1, 2 \ldots, n_1$) is the mass polar moment of inertia of the $1j$th concentrated element attached at the section $x_{1j}$ in the first step rod.

Using $\Theta_1(x)$ and the recurrence formula, Eq. (54), we can determine $\Theta_1(x)$ ($i = 2, 3, \ldots, n_1$). It is noted that all the expressions of $\Theta_1(x)$ include the same one unknown parameter, $M_1(0)$. Using $\Theta_1(x)$ and the boundary condition, Eq. (56), one obtains the frequency equation as

$$\begin{aligned}&\Theta_{n-1}(l_{n-1})\bar{S}_{n1}(l_n) + \frac{1}{GJ_n(0)}[M_{n-1}(l_{n-1}) - \omega^2 I_{n-1}\Theta_{n-1}(l_{n-1})]\bar{S}_{n2}(l_n)\\ &- \sum_{j=1}^{n_n}\frac{\omega^2 I_{nj}\Theta_n(x_{nj})}{GJ_n(x_{nj})}\bar{S}_{n2}(l_n - x_{nj}) = 0.\end{aligned} \tag{58}$$

(2) *Two free ends*: The boundary conditions for this case are

$$M_1(0) = 0, \tag{59}$$

$$M_n(l_n) = 0, \quad \text{i.e.,} \quad \Theta_n'(l_n) = 0. \tag{60}$$

Substituting Eq. (59) into Eq. (54), one obtains

$$\Theta_1(x) = \Theta_1(0)\bar{S}_{11}(x) - \sum_{j=1}^{n_1} \frac{\omega^2 I_{1j}\Theta_1(x_{1j})}{GJ_1(x_{1j})}\bar{S}_{12}(x - x_{1j})H(x - x_{1j}). \tag{61}$$

The unknown parameter in the above expression is $\Theta_1(0)$. As mentioned previously, all the expressions of $\Theta_1(x)$ $(i = 2, 3, \ldots n)$ include the same one unknown parameter as $\Theta_1(x)$ has. Hence, using one boundary condition at $x = l_n$, the frequency equation is obtained as

$$\begin{aligned} &\Theta_{n-1}(l_{n-1})\bar{S}'_{n1}(l_n) + \frac{1}{GJ_n(0)}[M_{n-1}(l_{n-1}) - \omega^2 I_{n-1}\Theta_{n-1}(l_{n-1})]\bar{S}'_{n2}(l_n) \\ &- \sum_{j=1}^{n_n} \frac{\omega^2 I_{nj}\Theta_n(x_{nj})}{GJ_n(x_{nj})}\bar{S}'_{n2}(l_n - x_{nj}) = 0. \end{aligned} \tag{62}$$

(3) *One fixed end* (*at* $x = 0$)–*one free end* (*at* $x = l_n$): Substituting the boundary condition at $x = 0$, Eq. (55), into Eq. (54), one obtains the expression of $\Theta_1(x)$, which has the same form as Eq. (57). Applying the boundary condition at $x = l_n$, Eq. (60), to $\Theta_1(x)$, one can establish the frequency equation which is the same as Eq. (62).

(4) *Two spring ends with concentrated elements*: When the left end ($x = 0$) and the right end ($x = l_n$) are spring supports with concentrated elements, the torsional stiffness and the mass polar moments of inertia of disks at $x = 0$ and $l_n$ are $K_0$, $I_0$ and $K_n$, $I_n$, respectively, the boundary conditions for this case are

$$M_1(0) = (K_0 - \omega^2 I_0)\Theta_1(0), \tag{63}$$

$$\Theta'_n(l_n) = -\frac{K_n - \omega^2 I_n}{GJ_n(l_n)}\Theta_n(l_n). \tag{64}$$

Substituting Eq. (63) into Eq. (54), one obtains

$$\Theta_1(x) = [\bar{S}_{11}(x) + (K_0 - \omega^2 I_0)\bar{S}_{12}(x)]\Theta_1(0) - \sum_{j=1}^{n_1} \frac{\omega^2 I_{1j}\Theta_1(x_{1j})}{GJ_1(x_{1j})}\bar{S}_{12}(x - x_{1j})\mathrm{H}(x - x_{1j}). \tag{65}$$

Using $\Theta_1(x)$ and the recurrence formula, one can obtain the expression of $\Theta_1(x)$. Applying the boundary condition at $x = l_n$, Eq. (64), to the expression of $\Theta_n(x)$ results in the frequency equation as

$$\begin{aligned} &\Theta_{n-1}(l_{n-1})\bar{S}'_{n1}(l_n) + \frac{1}{GJ_n(0)}\left[M_{n-1}(l_{n-1}) - \omega^2 I_{n-1}\Theta_{n-1}(l_{n-1})\right]\bar{S}'_{n2}(l_n) \\ &- \sum_{j=1}^{n_n} \frac{\omega^2 I_{nj}\Theta_n(x_{nj})}{GJ_n(x_{nj})}\bar{S}'_{n2}(l_n - x_{nj}) + \frac{K_n - \omega^2 I_n}{GJ_n(l_n)}\Bigg\{\Theta_{n-1}(l_n)\bar{S}_{n1}(l_n). \\ &+ \frac{1}{GJ_n(0)}[M_{n-1}(l_{n-1}) - \omega^2 I_{n-1}\Theta_{n-1}(l_{n-1})]\bar{S}_{n2}(l_n) \\ &- \sum_{j=1}^{n} \frac{\omega^2 I_{nj}\Theta_n(x_{nj})}{GJ_n(x_{nj})}\bar{S}_{n2}(l_n - x_{nj})\Bigg\} = 0. \end{aligned} \tag{66}$$

By the method of trial and error, one can obtain a set of natural frequencies, $\omega_j$, substituting $\omega_j$ into Eq. (54) the $j$th mode shape is determined.

## 3. Numerical example

The natural frequencies of torsional vibration of a five-step non-uniform rod with five rigid disks shown in Fig. 3 will be determined to demonstrate the application of the proposed method. The polar moment of inertia, the shear modulus, the mass intensity of the rod, the mass polar moment of inertia of the rigid disks and other parameters are given as

$$G = 8.3 \times 10^{10}\ \mathrm{N/m^2}$$

$$\rho = 7.8 \times 10^{6}\ \mathrm{kg/m^3}$$

$$J_i(x) = \alpha_i \mathrm{e}^{-\beta_i x/l_i}, \qquad \alpha_1 = 1.60 \times 10^{-8}\ \mathrm{m^4}, \qquad \alpha_2 = 1.38 \times 10^{-8}\ \mathrm{m^4},$$
$$\alpha_3 = 1.28 \times 10^{-8}\ \mathrm{m^4}, \qquad \alpha_4 = 1.20 \times 10^{-8}\ \mathrm{m^4}, \qquad \alpha_5 = 1.0 \times 10^{-8}\ \mathrm{m^4},$$

$$\beta_1 = 0.1, \qquad \beta_2 = 0.05, \qquad \beta_3 = 0, \qquad \beta_4 = 0.05, \qquad \beta_5 = 0.08,$$

$$l_i = 1\ \mathrm{m} \quad (i = 1, 2, 3, 4),$$

$$I_i = 0.32\ \mathrm{kg\,m^2} \quad (i = 1, 2, 3, 4, ), \qquad I_5 = 0.16\ \mathrm{kg\,m^2}.$$

The boundary conditions at $x = 0$ and $l_n$ are given in Eqs. (55) and (64), respectively, with $k_n = 0$ for this case. The mode shape function of the first step rod, $\Theta_1(x)$, is given in Eq. (57). Using $\Theta_1(x)$ and the recurrence formula, Eq. (54), we obtain $\Theta_1(x)$ $(i = 2, 3, 4, 5)$. Applying the boundary condition at $x = l_n$ to the expression of $\Theta_5(x)$, the frequency equation is established as

$$\Theta_4(l_4)\bar{S}'_{51}(l_5) + \frac{1}{GJ_5(0)}[M_4(l_4) - \omega^2 I_4 \Theta_4(l_4)]\bar{S}'_{52}(l_5)$$
$$- \frac{\omega^2 I_5}{GJ_5(l_5)}\left\{\Theta_4(l_4)\bar{S}_{51}(l_5) + \frac{1}{GJ_5(0)}[M_4(l_4) - \omega^2 I_4 \Theta_4(l_5)]\bar{S}_{51}(l_4)\right\} = 0.$$

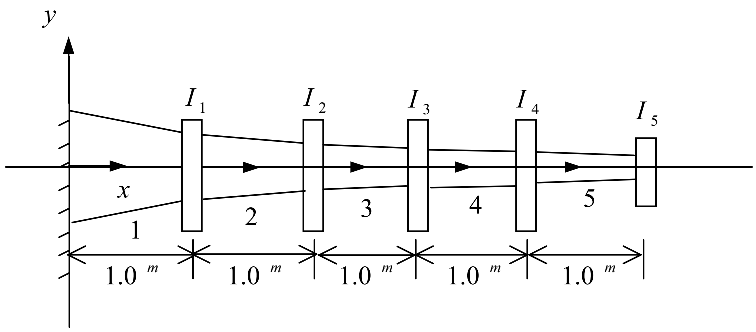


Fig. 3. A five-step non-uniform cantilever rod carrying five rigid disks.

Solving this equation by the method of trial and error obtains a set of $\omega_j$. The first three circular natural frequencies are listed as follows:

$$\omega_1 = 19.2908\,\text{rad/s}, \qquad \omega_2 = 57.7180\,\text{rad/s}, \qquad \omega_3 = 95.5542\,\text{rad/s}.$$

Substituting $\omega_j$ ($j = 1, 2, 3$) into Eq. (54) we obtain the $j$th mode shape ($j = 1, 2, 3$), which are shown in Table 1. If all the mass polar moments of the five rigid disks are distributed to the whole twisting rod, and the five-step rod is regarded as a one-step non-uniform rod with continuously distributed torsional stiffness and mass polar moment of inertia, then, we have

$$\begin{aligned} GJ(x) &= \alpha e^{-\beta x/L}, \quad \alpha = 1.330 \times 10^3\,\text{N m}^2, \quad \beta = 0.055, \\ I(x) &= a e^{-bx/L}, \quad a = 0.413\,\text{kg m}, \quad b = 0.03, \quad L = 5\,\text{m}. \end{aligned} \tag{67}$$

Substituting Eq. (67) into Eq. (17), one obtains

$$p(\varsigma) = a\alpha \left(\frac{L}{\alpha\beta}\right)^{(\alpha+\beta)/\beta} \varsigma^{-(\alpha+\beta)/\beta}. \tag{68}$$

The above expression is a special case of Eq.(19), then the frequency equation is

$$J_{-v}(\bar{\gamma})J_{v-1}(\bar{\gamma}A) = -J_v(\bar{\gamma})J_{-(v-1)}(\bar{\gamma}A), \tag{69}$$

where

$$\bar{\gamma}^2 = \frac{4a\omega^2 L^2}{\alpha f^2}, \qquad f = \beta - b, \qquad v = \frac{\beta}{\beta - b}, \qquad A = e^{f/2}. \tag{70}$$

Solving Eq. (69) and using Eq. (70), we obtain a set of $\omega_j$. The first three circular natural frequencies are given below

$$\omega_1' = 19.2910\,\text{rad}/s, \qquad \omega_2' = 57.7182\,\text{rad}/s, \qquad \omega_3' = 95.5547\,\text{rad}/s.$$

Table 1
The first three mode shapes of the five-step non-uniform cantilever rod carrying five rigid disks

| $x$ (m) | $X_1$ $(x)$ | | $X_2$ $(x)$ | | $X_3$ $(x)$ | |
|---|---|---|---|---|---|---|
| | Proposed method | FEM | Proposed method | FEM | Proposed method | FEM |
| 0.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.5 | 0.1426 | 0.1425 | 0.4437 | 0.4436 | 0.6907 | 0.6904 |
| 1.0 | 0.3250 | 0.3248 | 0.8123 | 0.8122 | 0.9983 | 0.9980 |
| 1.5 | 0.4454 | 0.4452 | 0.9693 | 0.9691 | 0.6913 | 0.6901 |
| 2.0 | 0.6028 | 0.6027 | 0.9545 | 0.9543 | −0.0046 | −0.0040 |
| 2.5 | 0.7025 | 0.7023 | 0.6946 | 0.6944 | −0.7015 | −0.7011 |
| 3.0 | 0.8253 | 0.8251 | 0.3228 | 0.3239 | −0.9991 | −0.9990 |
| 3.5 | 0.8883 | 0.8880 | −0.1606 | −0.1604 | −0.6950 | −0.6947 |
| 4.0 | 0.9694 | 0.9690 | −0.6096 | −0.6092 | 0.0098 | 0.0097 |
| 4.5 | 0.9771 | 0.9768 | −0.8814 | −0.8814 | 0.7110 | 0.7110 |
| 5.0 | 1.0000 | 1.0000 | −1.0000 | −1.0000 | 1.0000 | 1.0000 |

The FEM with linear approximation of displacement is also adopted to examine the accuracy of the proposed method. In the FEM analysis for this example, each step rod is divided into 20 elements, the first three circular natural frequencies are determined as

$$\omega_1'' = 19.2909\,\text{rad/s}, \quad \omega_2'' = 57.7181\,\text{rad/s}, \quad \omega_3'' = 95.5544\,\text{rad/s}.$$

The first three mode shapes obtained from the FEM are also shown in Table 1 for comparison purposes. It is evident that the results calculated by the two methods are in good agreement. However, it is revealed from our computation that the proposed method takes less computational time than FEM, thus illustrating the present method is efficient, convenient and accurate. The above calculated results also illustrate that a multi-step non-uniform rod carrying several rigid disks may be treated as a one-step non-uniform rod with continuously distributed torsional stiffness and mass polar moment of inertia for free vibration analysis.

## 4. Conclusions

An analytical approach and exact solutions for the torsional vibration of a multi-step non-uniform rod carrying an arbitrary number of concentrated elements and with classical or non-classical boundary conditions is proposed in this paper. The exact solutions for the free torsional vibration of non-uniform rods whose variations of cross-section are described by exponential functions and power functions are obtained. Then, the exact solutions for more general cases, non-uniform rods with arbitrary cross-section, are derived for the first time. Therefore, the analytical solutions obtained previously by Blevins [4], Pouyet and Lataillade [5], Li et al. [6], etc. for special types of non-uniform rods actually result as special cases of the present exact solutions for torsional vibration of non-uniform rods with arbitrary cross-section. The fundamental solutions and recurrence formulas are developed to simplify the analysis for the title problem. The mode shape functions of a multi-step non-uniform rod carrying various concentrated elements are easily expressed in terms of the fundamental solutions. The main advantage of the proposed methods is that the resulting frequency equation for torsional vibration of a multi-step non-uniform rod carrying arbitrary number of concentrated elements and with classical or non-classical boundary conditions is determined from a homogeneous algebraic equation with one unknown initial parameter. As a consequence, the computational time required by the proposed methods can be reduced significantly as compared with previously developed analytical procedures. A numerical example demonstrates that the results obtained from the proposed method are in good agreement with those determined from the FEM, but the proposed method takes less computational time than FEM, illustrating the present method is efficient, convenient and accurate. It is also shown through the numerical example that a multi-step non-uniform rod carrying several concentrated elements may be treated as a one-step non-uniform rod with continuously distributed torsional stiffness and mass polar moment of inertia for free vibration analysis.

## Acknowledgements

The author is thankful for the reviewers' useful comments and suggestions. The work described in this paper was fully supported by a grant from the City University of Hong Kong [Project Ref. No. 7001279].

## References

[1] G.W. Housner, T. Vreeland, Analytical Methods in Vibration, Macmillan, New York, 1967, pp. 156–161.
[2] G.Q. Li, Calculation Theory for Earthquake Resistance Structures, National Earthquake Press, Beijing, 1985, pp. 340–350.
[3] R.W. Clough, J. Penzien, Dynamics of Structures, McGraw-Hill, New York, 1993, pp. 370–380.
[4] R.D. Blevins, Formulas for Natural Frequency and Mode Shape, Van Nostrand Reinhold, New York, 1979, pp. 187–197.
[5] J.M. Pouyet, J.L. Lataillade, Torsional vibration of a shaft with non-uniform cross-section, Journal of Sound and Vibration 76 (1) (1981) 13–22.
[6] Q.S. Li, H. Cao, G.Q. Li, Static and dynamic analysis of straight bars with variable cross-section, Computers and Structures 59 (1996) 1185–1191.
[7] Q.S. Li, Classes of exact solutions for several static and dynamic problems of non-uniform beams, Structural Structures and Mechanics 12 (1) (2001) 85–100.
[8] Ö.S. Sener, H.N. Özgüven, Dynamic analysis of geared shaft systems by using a continuous system model, Journal of Sound and Vibration 166 (3) (1993) 539–556.
[9] D.B. Li, Y.H. Chui, I. Smith, Effect of warping on torsional vibration of members with open cross-sections, Journal of Sound and Vibration 170 (2) (1994) 270–275.
[10] E.J. Saapountzakis, Solutions of non-uniform torsion of bars by an integral equation method, Computers and Structures 77 (2000) 659–667.
[11] Q.S. Li, L.F. Yang, G.Q. Li, The quadratic finite element and strip with generalized degree of freedom and their application, Finite Element in Analysis and Design 34 (4) (2001) 325–339.